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Optimal mechanism design with resale via bargaining ☆

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Abstract

In this paper, we examine the optimal mechanism design of selling an indivisible object to one regular buyer and one publicly known buyer, where inter-buyer resale cannot be prohibited. The resale market is modeled as a stochastic ultimatum bargaining game between the two buyers. We fully characterize an optimal mechanism under general conditions. Surprisingly, in this optimal mechanism, the seller never allocates the object to the regular buyer regardless of his bargaining power in the resale market. The seller sells only to the publicly known buyer, and reveals no additional information to the resale market. The possibility of resale causes the seller to sometimes hold back the object, which under our setup is never optimal if resale is prohibited. We find that the seller's revenue is increasing in the publicly known buyer's bargaining power in the resale market. When the publicly known buyer has full bargaining power, Myerson's optimal revenue is achieved; when the publicly known buyer has no bargaining power, a conditionally efficient mechanism prevails.

JEL classification: C72; D44; D82; D83; L12

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1. Introduction

In the traditional mechanism design literature, buyers cannot resell the objects they just won. For example, in his seminal paper, Myerson [11] characterizes the optimal mechanism of selling an indivisible object to many privately informed buyers. In that mechanism, the buyer with the highest virtual valuation is awarded the object, providing that it is higher than the seller's reservation value. However, the Myerson allocation may be *ex-post* inefficient among the buyers if the buyers are asymmetric, as the highest virtual valuation buyer may not have the highest valuation. Therefore, buyers may be able to benefit from trading among themselves. Thus, when this kind of inter-buyer resale cannot be prohibited, buyers will engage in resale upon the Myerson allocation. Given this, the final allocation of the object rolls away from the Myerson allocation, providing different incentives for the buyers in the mechanism. As such, the Myerson optimal revenue may not be achievable.

Many researchers have started to address this issue of resale in auctions and optimal mechanisms. These include Ausubel and Cramton [1], Calzolari and Pavan [2], Cheng and Tan [3], Garratt and Troger [4], Hafalir and Krishna [5], Haile [6], Virag [14], and Zheng [15], all of which we will discuss in this introduction. Markedly, Zheng [15] demonstrates that even if the seller cannot prohibit resale, she can still achieve the Myerson revenue under certain resale rules. He constructs a mechanism involving many rounds of resales, with the winner of each round reselling the object to the rest of the buyers, leading to the Myerson allocation in the end. The Revenue Equivalence Theorem implies that this mechanism generates the same revenue as the Myerson revenue, which is the upper bound revenue among all feasible mechanisms. (Note that from the Revenue Equivalence Theorem the seller cannot earn more revenue by allowing for resales.) As a result, Zheng's mechanism is optimal among all mechanisms with resale. One key feature in Zheng's mechanism is that the winner in each round has full bargaining power, dispensing a mechanism that is optimal for himself. In the case where the winner of each round has less than full bargaining power, Myerson's revenue may no longer be achievable.[1] In this case, Zheng's construction does not apply. We characterize an optimal mechanism when the winner has less than full bargaining power in this paper.

Our approach is to analyze the buyers' incentive compatibility constraints and participation constraints directly. We adopt the framework of Garratt and Troger [4] in our analysis. Even though the framework is used by them to study the equilibrium behavior in certain auctions with resale, we find it suitable for examining the optimal mechanism design problem as well. In our model, in addition to the seller, there are two buyers: one regular buyer, and one publicly known buyer whose valuation is commonly known. This publicly known buyer resembles a speculator or a dealer. The two buyers can engage in resale activities. The seller cannot control what happens in the resale market.

Departing from Zheng's and Garratt and Troger's assumption that the winner has full bargaining power, we model the resale market as a stochastic ultimatum bargaining game between the two buyers. With certain probability, the winner is picked as the proposer in the ultimatum

[1] As we shall show in this paper, when resale is permitted, the Myerson revenue can be achieved when and only when the winner has full bargaining power or when the publicly known buyer's valuation is equal to the seller's.

bargaining game, and with the rest of the probability the loser is picked as the proposer. These probabilities can vary depending on who initially wins the object, and serve as the bargaining powers of the respective buyers.

In our analysis, the seller faces a mechanism design problem with hidden information, hidden actions and multiple agents, since the regular buyer's valuation is his private information and the buyers' actions in the resale market are not dictated by the seller. McAfee and McMillan [9] consider the optimal design of team mechanisms when players have privately known abilities and their individual efforts in the team production are unobservable. They find that the designer can achieve the same revenue as if the moral hazard problem were not present. In our paper, the fact that the seller has no control over the resale market activities reduces her rent extraction ability. Myerson [12] establishes the revelation principle and formulates mechanisms under this general setting, even though explicitly characterizing the optimal mechanism is nontrivial.[2]

We are able to explicitly characterize an optimal mechanism under general conditions. The most striking result is that the seller never allocates the object to the regular buyer directly in the initial market even when he has bargaining power in the resale market. The seller allocates the object to the publicly known buyer if his resale augmented virtual valuation is higher than the seller's reservation value (which is normalized to zero). The seller charges the publicly known buyer an amount equal to his expected benefit from resale, leaving him with zero expected total surplus. Although the seller never allocates the object to the regular buyer directly, she nevertheless demands some payment from the regular buyer if she decides not to retain the object. More importantly, the seller reveals no private information to the resale market; the buyers only know who just won the object when entering the resale market.

It turns out that the buyers' bargaining powers in the resale market determine crucially the revenue the seller can optimally achieve. In general, the seller's maximal revenue with resale is less than the maximal revenue with no resale (i.e., the Myerson revenue), since the seller has more controlling power in the case of no resale. We show that the Myerson revenue can be achieved only when the publicly known buyer has full bargaining power or has zero valuation. If the regular buyer has full bargaining power, a conditionally efficient allocation is optimal. In this conditionally efficient allocation, the allocation among the buyers is always efficient, but the seller may retain the object inefficiently. In fact, the seller's revenue is an increasing function of the publicly known buyer's bargaining power. When the regular buyer's augmented virtual valuation is always greater than the seller's reservation value, the seller's revenue is a weighted average of the Myerson revenue and the fully efficient allocation revenue, with weights equal to the players' bargaining powers.

In our model, the publicly known buyer's virtual valuation is always greater than the seller's reservation value, and therefore the seller never retains the object in the Myerson allocation. But with resale, it may be optimal for the seller to retain the object under some circumstances. As an implication, resale induces a more efficient allocation among the buyers, but at the same time introduces a new source of inefficiency. Therefore, whether resale can improve the overall efficiency in the optimal mechanism is ambiguous.

The literature on auctions with resale has provided us with significant insights. Hafalir and Krishna [5], for example, examine the first and second price auctions with possible resale. There are two players, and either the winner or the loser in the auction has the chance to make a

[2] In the literature of optimal incentive contracts, McAfee and McMillan [8], and Laffont and Tirole [7] characterize the optimal contract for a principal facing agents with privately known abilities, unobservable efforts, but observable outputs. In our model, the seller has no control of the resale market except deciding on how much information to reveal.

take-it-or-leave-it offer to the other. They find that the two players, although asymmetric, win with equal probability in the auctions. They also find that first price auctions generate more revenue than second price auctions. Later, Virag [14] extends their analysis to the case of many players who can be classified into two groups: weak and strong. He shows that with more than two players, strong players win more often than weak players. Recently, Cheng and Tan [3] study the asymmetric common-value auctions and apply the results to the revenue ranking in independent private value auctions with resale. In doing so, they generalize the analysis in Hafalir and Krishna [5] by relaxing their regularity assumption and find that the revenue ranking in Hafalir and Krishna [5] could sometimes be reversed. Given resale opportunities, the issue of publicly known buyers has also been studied. Garratt and Troger [4] consider the first and second price auctions with many symmetric bidders and an additional speculator whose valuation is commonly known to be zero. In the resale market, the winner may use a standard auction with an optimal reserve price as well as an optimal mechanism to resell the object. They find that the speculator can play an active role in the equilibrium.[3]

In this paper, we use a mechanism design approach to characterize an optimal mechanism with resale. In the process, we address one important issue that is not addressed in any of the above papers, that is, how should the seller control the amount of information revealed to the resale market? The common assumption in the literature is that only the transaction price (i.e., the highest bid in a first price auction and the second highest bid in a second price auction) is announced by the seller. The question remains whether this is optimal for the seller to do so. In theory, the seller can have many different options. She can conceal all the information, reveal all the information, reveal the information stochastically or partially, etc. Obviously, it is almost impossible to formulate all possible announcement rules one by one. Our paper takes one step further by considering the optimal rule for information revelation. In the optimal mechanism we constructed, concealing all the information is the rule.

Our paper is closest to Calzolari and Pavan [2], who consider the issue of information transmission to the resale market in their optimal mechanism. They mainly focus on the case of reselling the object to a third party. In the case of inter-bidder resale, they assert that any deterministic mechanism cannot be optimal. With two bidders and two-point valuation distributions, they provide a characterization of the optimal mechanism in the case where one of the bidders has full bargaining power (in their online appendix). In our model, the regular buyer has a continuous valuation distribution, while the publicly known buyer has a commonly known valuation. Furthermore, we allow the buyers to have partial bargaining powers. Because of these partial bargaining powers and the continuous distribution of one buyer's valuation, we use a different method to characterize the optimal mechanism. One result that we obtain is that the seller's revenue is an increasing function of the bargaining power of the initial winner (who always turns out to be the publicly known buyer).

Our paper is also related to Ausubel and Cramton [1], who characterize the optimal mechanism when the resale market is perfect in the sense that any inefficiency will be corrected in the resale market. Given such a perfect resale market, they find that the seller should induce an efficient allocation directly in the initial market. However, the question remains how a perfect resale market can be constructed under asymmetric information.[4] Our analysis supports the optimality

[3] In all of these papers, resale arises because of the inefficiencies of the auction allocations. This is also the case in our paper. Haile [6] considers a different source of resale: after the auctions, new information becomes available and it alters the buyers' valuations.

[4] Myerson and Satterthwaite [13] show that inefficiency cannot necessarily be corrected in such a secondary market.

of the initial efficient allocation under certain conditions. If the initial loser has full bargaining power in the resale market, and the regular buyer's virtual valuation is always greater than the seller's reservation value, then it is optimal to allocate the object efficiently in the initial market.

Finally, our paper is related to Molnar and Virag [10], who characterize the optimal mechanism when the winning bidder can participate a post-auction market under asymmetric information. A bidder's payoff depends not only on his own valuation but also on his perceived type due to the post-auction market interactions. They find that the optimality of revealing or concealing information depends on the concavity of the payoff function over the signaled types. In our paper, concealing information is always optimal in the optimal mechanism we characterized.[5]

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we analyze the model and establish some incentive compatibility conditions for the resale market. In Section 4, we analyze the initial market and establish some incentive compatibility conditions for the entire game. In Section 5, we characterize an optimal mechanism for the seller. In Section 6, we conclude. All lengthy proofs are relegated to Appendix A.

2. The model

One seller (she) with one indivisible object faces two buyers. Buyer 1 has a commonly known valuation $v_1 \geqslant 0$ for the object. Buyer 2's valuation, v_2, is his private information. Assume that v_2 follows a distribution with $c.d.f.\ F(\cdot)$, $p.d.f.\ f(\cdot)$, and support $[a, b]$. We call buyer 2 the *regular buyer* and buyer 1 the *publicly known buyer*.[6]

Assume that the hazard rate, $\frac{f(v_2)}{1-F(v_2)}$, is increasing; this is a common assumption to simplify the characterization of the optimal mechanism. Let $J(v_2) = v_2 - \frac{1-F(v_2)}{f(v_2)}$ denote buyer 2's virtual valuation function. Note that buyer 1's valuation is also his virtual valuation, since it involves no uncertainty. Note also that $J(b) = b$. In this paper, without loss of generality, we normalize the seller's reservation value of the object to zero.

When the seller has full controlling power and can prohibit resales among the buyers, Myerson's optimal auction yields the highest revenue for the seller. In that optimal auction, the seller should allocate the object to the buyer with the higher nonnegative virtual valuation. Since the publicly known buyer's virtual valuation (i.e., his valuation) is nonnegative in our model, the seller does not retain the object in Myerson's optimal auction. Furthermore, if buyer 1's valuation is always less than or equal to buyer 2's virtual valuation, i.e. $v_1 \leqslant J(a)$, then the seller should always allocate the object to buyer 2. On the other hand, if buyer 1's valuation is always greater than or equal to buyer 2's virtual valuation, i.e., $v_1 \geqslant J(b) = b$, then the seller should always allocate the object to buyer 1. In the former case, $v_1 \leqslant J(a)$ implies that $v_1 \leqslant v_2$, $\forall v_2 \in [a, b]$. In the latter case, $v_1 \geqslant b$ implies that $v_1 \geqslant v_2$, $\forall v_2 \in [a, b]$. In these two cases, since the buyer with the higher valuation receives the object, resale (if allowed) will not happen, and the object allocation is efficient.

The above observation leads to the following proposition: given those conditions above, there will be no resale given the Myerson allocations, and thus the Myerson revenue (i.e., the maximum revenue) is achieved even if resale is allowed.

[5] In a working paper version of Molnar and Virag [10], the post-auction market interactions are either a Cournot competition or a Bertrand competition. Therefore, similarly to our model, the seller has no control over the firms' production or pricing decisions in the post-auction market.

[6] When buyer 1's valuation is zero, he is called a speculator in Garratt and Troger [4]. Here, we allow for more general speculators, with valuations that could be greater than zero. This publicly known buyer can also be thought of as a dealer.

Proposition 1. *Suppose that resale between buyers is allowed. If* $v_1 \leqslant J(a)$ *or* $v_1 \geqslant b$*, then the seller can achieve the highest revenue by implementing the Myerson allocation: always assign the object to buyer* 2 *if* $v_1 \leqslant J(a)$*, and always assign the object to buyer* 1 *if* $v_1 \geqslant b$*. The allocation is ex-post efficient and no resale between the two buyers occurs.*

In what follows, we will assume that buyer 1's valuation lies inside the range of buyer 2's virtual valuations.

Assumption 1. $J(a) < v_1 < b$.

When Assumption 1 holds, the Myerson allocation is sometimes ex-post inefficient, as buyer 1's valuation is sometimes higher than buyer 2's virtual valuation but lower than buyer 2's valuation. In this case, there is strictly positive incentive for buyer 1 to resell the object to buyer 2. The objective of this paper is to characterize the optimal mechanism when this kind of resale is allowed (or equivalently, when resale cannot be prohibited by the seller).

Formally, we model this sale with possible resale situation as follows. There are two markets: the initial market and the resale market. The initial market resembles a commonly used mechanism. The seller decides the allocation of the object and the transfer payments. The resale market is modeled as a stochastic ultimatum bargaining game and nature randomly picks a buyer as the proposer with certain probability. We allow the probability to depend on who owns the object when entering the resale market (i.e., who won the object in the initial market). To be more concrete, we assume that when the publicly known buyer owns the object, with probability λ_1, the publicly known buyer is picked to propose a take-it-or-leave-it offer to the regular buyer, and the regular buyer chooses to either accept or reject the offer; with probability $1 - \lambda_1$, the regular buyer is picked to propose and the procedure is similar. When the regular buyer owns the object, with probability λ_2, the publicly known buyer is picked, and with probability $1 - \lambda_2$, the regular buyer is picked. Transaction takes place if the proposed offer is accepted. There are no further rounds of bargaining if the proposed offer is rejected. Note that λ_1 and λ_2 capture the publicly known buyer's bargaining powers in each situation.

The seller designs a mechanism to sell the object. If the seller decides to retain the object in the initial market, the game ends; otherwise, the two buyers enter the resale market. The seller has no control over the resale market, i.e., the structure of the resale market is exogenously given and the seller cannot force the buyers to act in any way in the resale market.[7] In the initial market, the seller can decide on the object allocation and the monetary transfers. In addition, the seller can decide on how much information to reveal in an attempt to influence the buyers' actions in the resale market.

In this dynamic mechanism design problem, the seller plays an active role in information revelation. After the seller sees the information (i.e., reports) from the buyers, she can decide on what information to reveal to the buyers. This information could affect the prices in the resale market, and therefore could affect the buyers' behaviors in the initial market. This effect is not present in a static mechanism design problem. In this paper, we assume that the seller has full control over this revelation of information, which is costless to her.

The seller in our model needs to design an optimal mechanism taking into consideration the buyers' resale behaviors. The buyers' behaviors in the resale market are not contractible by the

[7] If the seller has full control of the buyers' actions in the resale market, then we are back to the situation of Myerson [11].

seller. This is a mechanism design problem with hidden information, hidden actions and multiple agents. The challenge is the seller not only deciding the object allocations and monetary transfers, but also controlling the information revealed to the resale market.

We make use of the revelation principles in Myerson [12] throughout our analysis and restrict our search of the optimal mechanism to direct mechanisms without loss of generality. This revelation principle originally deals with discrete types, but can be extended to continuous types by changing summations to integrals in the derivations. In a direct mechanism, the buyers first report their types to the seller; the seller then decides on the object allocation and monetary transfers, as well as the confidential recommendations sent to the buyers regarding their actions in the resale market.[8]

In a direct mechanism of our model, buyer 1 (the publicly known buyer) does not have any private information and thus has nothing to report. Buyer 2 confidentially sends a report of his valuation $\tilde{v}_2$ to the seller. Then with probabilities $x_1(\tilde{v}_2)$ and $x_2(\tilde{v}_2)$, the seller allocates the object to buyer 1 and buyer 2, respectively. In addition, depending on the report $\tilde{v}_2$, the seller sends each buyer a confidential recommendation on what action the buyer should take in the resale market.

The seller' recommendation to a buyer depends on whether he is allocated the object. Furthermore, a recommendation is contingent on whether or not the buyer is the one to make the take-it-or-leave-it offer in the resale market. Therefore, there are four different situations, indexed by who wins and who is picked to make the offer in the resale market. In what follows, we index these situations by **Case** ij, in which buyer i is allocated the object in the initial market and buyer j is selected to make the offer in the resale market, $i, j \in 1, 2$.

In our analysis, the seller does not send acceptance recommendations to the buyers, because the only sequentially rational action is for the initial winner (the buyer who is allocated the object in the initial market) to accept an offer if the price is higher than or equal to his valuation and for the initial loser (the buyer who is not allocated the object in the initial market) to accept an offer if the price is lower than or equal to his valuation.

Let $p_{ij}(\tilde{v}_2)$ denote the price that the seller recommends buyer j to offer to the other buyer in **Case** ij. The seller can also randomize her recommendations given $\tilde{v}_2$; in this case $p_{ij}(\tilde{v}_2)$ denote the realization of the randomization.[9] Since $p_{ij}(\tilde{v}_2)$ depends on buyer 2's report $\tilde{v}_2$, buyer 1 may be able to infer from it some information regarding $\tilde{v}_2$.

In our model, the seller sends out her confidential recommendations after the object allocation is realized in the initial market but before the buyers learn who is selected to propose the price in the resale market. This is because the random selection of proposers is just a way of modeling the buyers' bargaining power in the resale market, in which the seller has no control. Therefore, when buyer i wins in the initial market, he privately learns the recommendation $p_{ii}(\tilde{v}_2)$. Meanwhile, buyer $j \neq i$ privately learns the recommendation $p_{ij}(\tilde{v}_2)$.[10] Because the recommendations

[8] Our methodology is similar to Calzolari and Pavan [2], where the revelation principle in Myerson [12] was utilized to analyze a model of similar structure to ours.

[9] Here, we do not consider all stochastic mechanisms which, in the most generous case, allow the allocation rules and the recommendations to be correlated with each other conditional on $\tilde{v}_2$.

[10] If the seller cannot make her recommendations conditioning on who wins, then she sends $p_{11}(\tilde{v}_2)$ and $p_{21}(\tilde{v}_2)$ to buyer 1 and $p_{12}(\tilde{v}_2)$ and $p_{22}(\tilde{v}_2)$ to buyer 2. Obviously, under this setting, the seller cannot generate more revenue. However, as we shall show in the analysis, the recommendations in our optimal mechanism do not depend on who wins anyway. This means that the optimal mechanism characterized in this paper remains optimal under this alternative setup.

contain information about buyer 2's private report of his valuation (which becomes a truthful report in equilibrium), buyer 1 can update his belief about buyer 2's valuation upon receiving the recommendation.

In a direct mechanism, the monetary transfers $t_1(\tilde{v}_2)$ and $t_2(\tilde{v}_2)$ from buyers 1 and 2, respectively, to the seller are collected privately at the very end after the resale market is closed. In an indirect mechanism, the monetary transfers can be collected at any time. If the buyers learn about the monetary transfers before the resale market is open, then the buyers can update their beliefs accordingly. However, such an indirect mechanism has a counterpart in the direct mechanisms; any information revealed by the monetary transfers can be conveyed by the seller's recommendations as well. Of course, if a buyer's monetary transfer does not depend on the other buyer's valuation report, then it does not need to be collected at the end, since the transfer does not reveal any private information of the other buyer. As we shall show in later analysis, this is indeed the case in the optimal mechanism we characterize.

The seller maximizes her revenue by selecting the allocation rules, $x_1(\tilde{v}_2)$ and $x_2(\tilde{v}_2)$, recommendations, $p_{11}(\tilde{v}_2)$, $p_{12}(\tilde{v}_2)$, $p_{21}(\tilde{v}_2)$, $p_{22}(\tilde{v}_2)$, and monetary transfers, $t_1(\tilde{v}_2)$ and $t_2(\tilde{v}_2)$, subject to the incentive compatibility constraints, the participation constraints, and the feasibility constraints. Note that only buyer 2 has private information. The incentive compatibility constraint for buyer 1 (IC_1^R) is that he will optimally follow the recommendations in the resale market, given that buyer 2 truthfully reports his valuation in the initial market and follows the recommendations in the resale market.[11] The incentive compatibility constraint for buyer 2 is that he will report his valuation truthfully in the initial market and follow the recommendations in the resale market, given that buyer 1 follows the recommendations in the resale market. We break up buyer 2's incentive compatibility constraints into two parts. The first part (IC_2^R) is that, if buyer 2 has truthfully reported his valuation in the initial market, it is optimal for him to follow the seller's recommendations in the resale market. The second part (IC_2^I) is that, buyer 2 will truthfully report his valuation in the initial market given that he will behave optimally in the resale market. The participation constraints for the buyers (PC_1 and PC_2) require that participating in the mechanism is better than their outside options, which are normalized to zero here. Since there is only one object to be allocated, the following feasibility constraint must be satisfied:

$$x_1(v_2) \geqslant 0, \qquad x_2(v_2) \geqslant 0, \qquad x_1(v_2) + x_2(v_2) \leqslant 1, \quad \forall v_2. \tag{1}$$

To summarize, the mechanism design problem for the seller is

$$\max \quad R = \int_a^b t_1(v_2)\, dF(v_2) + \int_a^b t_2(v_2)\, dF(v_2)$$

$$\text{subject to:} \quad IC_1^R,\ IC_2^R,\ IC_2^I,\ PC_1,\ PC_2,\ \text{and (1)}.$$

In the following sections, we will examine these constraints one by one, starting backward from the resale market.

3. The resale market: Establishing IC_1^R and IC_2^R

In this section, we examine the buyers' strategies in the resale market (the continuation game) on and off the equilibrium path. When buyer 2 has truthfully reported his valuation v_2 to the

11 Buyer 1 has no incentive compatibility constraint in the initial market, since he has no private information.

seller in the initial market, the incentive compatibility constraints require that both buyers should follow the seller's recommendations in the resale market. When buyer 2 lies about his valuation, buyer 1 still follows the seller's recommendation, while buyer 2 may act differently. In what follows, we examine the buyers' optimization problems and the related incentive compatibility constraints case by case.

Case 11: In this case, buyer 1 wins in the initial market and is also picked to make the offer in the resale market. The information buyer 1 has when determining the offer price is that he won and received recommendation p_{11}^*. Thus, buyer 1 chooses his offer price $\tilde{p}$ to maximize his payoff:

$$\max_{\tilde{p}} \quad v_1 \mathit{Prob}\{v_2 < \tilde{p} | \text{buyer 1 wins, } p_{11}(v_2) = p_{11}^*\} + \tilde{p}\, \mathit{Prob}\{v_2 \geqslant \tilde{p} | \text{buyer 1 wins, } p_{11}(v_2) = p_{11}^*\}. \tag{2}$$

Let

$$G_{11}(v_2) = F(v_2 | \text{buyer 1 wins, } p_{11}(v_2) = p_{11}^*),$$

and

$$g_{11}(v_2) = f(v_2 | \text{buyer 1 wins, } p_{11}(v_2) = p_{11}^*).$$

Then the above maximization problem can be written as

$$\max_{\tilde{p}} \quad \Pi_1 = v_1 G_{11}(\tilde{p}) + \tilde{p}[1 - G_{11}(\tilde{p})]. \tag{3}$$

The incentive compatibility constraint then implies that it is optimal for buyer 1 to follow the seller's recommendation; i.e., $\tilde{p} = p_{11}^*$. This is summarized in the following lemma.

Lemma 1. *In Case* 11*, the offer price $p_{11}(v_2)$ is incentive compatible if and only if*

$$p_{11}(v_2) \in \arg\max_{\tilde{p}} \{v_1 G_{11}(\tilde{p}) + \tilde{p}[1 - G_{11}(\tilde{p})]\}, \quad \forall v_2.$$

When the seller receives report $\tilde{v}_2$ and recommends $p_{11}(\tilde{v}_2)$ to buyer 1*, buyer* 1 *follows the recommendation and sets the price equal to $p_{11}(\tilde{v}_2)$. Resale occurs at that price if $p_{11}(\tilde{v}_2) \leqslant v_2$, and does not occur if $p_{11}(v_2) > v_2$.*

Note that the price $p_{11}(\tilde{v}_2)$ is never less than v_1. This is because buyer 1 will never ask for less than his valuation in the bargaining.

Case 12: In this case, buyer 1 wins in the initial market but buyer 2 is picked to make the offer in the resale market. Suppose that buyer 2 reports $\tilde{v}_2$ to the seller. The information buyer 2 has when determining the offer price is that he has lost in the initial market and received recommendation p_{12}^*. Since v_1 is common knowledge, buyer 2 optimally offers

$$p_{12}(v_2) = \begin{cases} v_1, & \text{if } v_2 \geqslant v_1; \\ \text{any distribution for price lower than } v_2, & \text{if } v_2 < v_1. \end{cases} \tag{4}$$

In this case, the offer price does not depend on buyer 2's own report to the seller. This is summarized in the following lemma.

Lemma 2. *In Case* 12*, the price offer is incentive compatible if and only if it satisfies Eq.* (4)*. When the seller receives report* $\tilde{v}_2$ *and recommends* $p_{12}(\tilde{v}_2)$ *to buyer* 2*, buyer* 2 *follows* $p_{12}(v_2)$*. Resale occurs at price* v_1 *if* $v_2 \geqslant v_1$*, and does not occur if* $v_2 < v_1$*.*

Case 22: In this case, buyer 2 wins in the initial market and is picked to make the offer in the resale market. Suppose that buyer 2 reports $\tilde{v}_2$ to the seller. The information buyer 2 has when deciding the offer price is that he has won in the initial market and received recommendations p^*_{22}. Similarly to Case 12, since v_1 is common knowledge, buyer 2 optimally offers

$$p_{22}(v_2) = \begin{cases} v_1, & \text{if } v_2 \leqslant v_1; \\ \text{any distribution for price higher than } v_2, & \text{if } v_2 > v_1. \end{cases} \tag{5}$$

Again, the prices do not depend on $\tilde{v}_2$. Note that in this case, buyer 2 is the seller in the resale market, while buyer 2 is the buyer in Case 12. This is summarized in the following lemma.

Lemma 3. *In Case* 22*, the offer price is incentive compatible if and only if it satisfies Eq.* (5)*. When the seller receives report* $\tilde{v}_2$ *and recommends* $p_{22}(\tilde{v}_2)$ *to buyer* 2*, buyer* 2 *follows* $p_{22}(v_2)$*. Resale occurs at price* v_1 *if* $v_2 \leqslant v_1$*, and does not occur if* $v_2 > v_1$*.*

Case 21: In this case, buyer 2 wins in the initial market and buyer 1 is picked to make the offer in the resale market. Suppose that buyer 2 reports $\tilde{v}_2$ to the seller. The information buyer 1 has when determining the offer price is that he has lost in the initial market and received recommendation p^*_{21}. Therefore, he chooses $\tilde{p}$ to maximize:

$$\max_{\tilde{p}} \quad (v_1 - \tilde{p}) Prob\big\{v_2 \leqslant \tilde{p} \,|\, \text{buyer 2 wins}, \; p_{21}(v_2) = p^*_{21}\big\}. \tag{6}$$

Let

$$G_{21}(v_2) = F\big(v_2 \,|\, \text{buyer 2 wins}, \; p_{21}(v_2) = p^*_{21}\big),$$

and

$$g_{21}(v_2) = f\big(v_2 \,|\, \text{buyer 2 wins}, \; p_{21}(v_2) = p^*_{21}\big).$$

Then the above maximization problem is equivalent to

$$\max_{\tilde{p}} \quad \Pi_{21} = (v_1 - \tilde{p}) G_{21}(\tilde{p}). \tag{7}$$

The incentive compatibility constraint induces buyer 1 to optimally follow the seller's recommendation p^*_{21} if $\tilde{v}_2 = v_2$. This is summarized in the following lemma.

Lemma 4. *In Case* 21*, the offer price is incentive compatible if and only if*

$$p_{21}(v_2) \in \arg\max_{\tilde{p}} \big\{(v_1 - \tilde{p}) G_{21}(\tilde{p})\big\}, \quad \forall v_2.$$

When the seller receives report $\tilde{v}_2$ *and recommends* $p_{21}(\tilde{v}_2)$ *to buyer* 1*, buyer* 2 *follows it. Resale occurs at price* $p_{21}(\tilde{v}_2)$ *if* $p_{21}(\tilde{v}_2) > v_2$*, and does not occur if* $p_{21}(\tilde{v}_2) \leqslant v_2$*.*

Again, $p_{21}(\tilde{v}_2)$ is never less than v_1 because buyer 1 would never ask for a price lower than his valuation.

4. The initial market: Establishing IC_2^I, PC_1 and PC_2

In the above section, Lemmas 1–4 characterize the recommendations that are incentive compatible in the resale market and the behavior of the buyers on and off the equilibrium path. In the following calculations, we will plug in those recommendations whenever possible, except $p_{11}(v_2)$ and $p_{21}(v_2)$ (which have no explicit solutions).

Since buyer 1 has no private information, he has no incentive compatibility constraint in the initial market. Suppose that buyer 2 truthfully reports his valuation and also follows the recommendation in the resale market. We can calculate buyer 1's total payoff when he always follows the seller's recommendations in the resale market:

$$U_1 = \int_a^b \mathbb{E}_{p_{11}(v_2), p_{21}(v_2)} \big\{ x_1(v_2) \big\{ \lambda_1 \big[v_1 I_{\{v_2 < p_{11}(v_2)\}} + p_{11}(v_2) I_{\{v_2 \geqslant p_{11}(v_2)\}} \big] + (1-\lambda_1) v_1 \big\} + x_2(v_2) \big\{ \lambda_2 \big(v_1 - p_{21}(v_2) \big) I_{\{v_2 \leqslant p_{21}(v_2)\}} \big\} - t_1(v_2) \big\} \, dF(v_2), \tag{8}$$

where $I_{\{.\}}$ is the indicator function, and the expectation $\mathbb{E}$ is taken because $p_{11}(\cdot)$, $p_{21}(\cdot)$ are potentially mixed actions. This calculation follows directly from the outcomes in the four cases in the on-equilibrium-path continuation games in the resale market as described in Lemmas 1–4. Rewriting Eq. (8), we can obtain the following lemma, which will be useful when formulating the seller's revenue.

Lemma 5. *Buyer* 1*'s expected transfer to the seller is given by*

$$\int_a^b t_1(v_2) \, dF(v_2) = \int_a^b \mathbb{E}_{p_{11}(v_2), p_{21}(v_2)} \big\{ x_1(v_2) \big\{ \lambda_1 \big[v_1 I_{\{v_2 < p_{11}(v_2)\}} + p_{11}(v_2) I_{\{v_2 \geqslant p_{11}(v_2)\}} \big] + (1-\lambda_1) v_1 \big\} + x_2(v_2) \big\{ \lambda_2 \big(v_1 - p_{21}(v_2) \big) I_{\{v_2 \leqslant p_{21}(v_2)\}} \big\} \big\} \, dF(v_2) + U_1. \tag{9}$$

We normalize buyer 1's payoff of not participating to zero. Then his participating constraint becomes

$$(PC_1) \quad U_1 \geqslant 0. \tag{10}$$

Note that (PC_1) should be binding in the optimal mechanism which maximizes the seller's revenue; if $U_1 > 0$, the seller can obtain a higher revenue by increasing $t_1(v_2)$ while keeping other terms unchanged.

Now suppose that buyer 1 always follows the seller's recommendation. The payoff for buyer 2 if he reports $\tilde{v}_2$ as his valuation (and subsequently acts optimally in the resale market, cf. Lemmas 1–4) is given by

$$U_2(v_2, \tilde{v}_2) = \mathbb{E}_{p_{11}(\tilde{v}_2), p_{21}(\tilde{v}_2)} \big\{ x_1(\tilde{v}_2) \big\{ \lambda_1 \big[v_2 - p_{11}(\tilde{v}_2) \big] I_{\{v_2 \geqslant p_{11}(\tilde{v}_2)\}} + (1-\lambda_1)(v_2 - v_1) I_{\{v_2 \geqslant v_1\}} \big\} + x_2(\tilde{v}_2) \big\{ (1-\lambda_2) [v_2 I_{\{v_2 > v_1\}} + v_1 I_{\{v_2 \leqslant v_1\}}] + \lambda_2 \big[p_{21}(\tilde{v}_2) I_{\{v_2 \leqslant p_{21}(\tilde{v}_2)\}} + v_2 I_{\{v_2 > p_{21}(\tilde{v}_2)\}} \big] \big\} - t_2(\tilde{v}_2) \big\}. \tag{11}$$

The formula follows directly from the four cases in the off-equilibrium-path continuation games in the resale market described in Lemmas 1–4. The incentive compatibility constraint and the participation constraint for buyer 2 are

$$(IC_2^I) \quad U_2(v_2, v_2) \geqslant U_2(v_2, \tilde{v}_2), \quad \forall v_2, \tilde{v}_2, \tag{12}$$

$$(PC_2) \quad U_2(v_2, v_2) \geqslant 0, \quad \forall v_2. \tag{13}$$

In the following analysis, following the convention of the mechanism design literature, we first replace IC_2^I by the first order condition of the maximization of buyer 2's payoff (11), and then prove that the derived optimal mechanism satisfies IC_2^I. We have the following lemma.

Lemma 6. *Buyer* 2*'s incentive compatibility constraint and participation constraint in the initial market are satisfied only if the following conditions hold*:

$$\begin{aligned} t_2(v_2) = \mathbb{E}_{p_{11}(v_2), p_{21}(v_2)} \Bigg\{ & x_1(v_2)\big\{\lambda_1\big[v_2 - p_{11}(v_2)\big] I_{\{v_2 \geqslant p_{11}(v_2)\}} \\ & + (1-\lambda_1)(v_2 - v_1) I_{\{v_2 \geqslant v_1\}}\big\} + x_2(v_2)\big\{(1-\lambda_2)[v_2 I_{\{v_2 > v_1\}} + v_1 I_{\{v_2 \leqslant v_1\}}] \\ & + \lambda_2\big[p_{21}(v_2) I_{\{v_2 \leqslant p_{21}(v_2)\}} + v_2 I_{\{v_2 > p_{21}(v_2)\}}\big]\big\} \\ & - \int_a^{v_2} \big\{x_1(\xi)\big[\lambda_1 I_{\{\xi \geqslant p_{11}(\xi)\}} + (1-\lambda_1) I_{\{\xi \geqslant v_1\}}\big] \\ & + x_2(\xi)\big[(1-\lambda_2) I_{\{\xi > v_1\}} + \lambda_2 I_{\{\xi > p_{21}(\xi)\}}\big]\big\} d\xi \Bigg\} - U_2(a, a), \end{aligned} \tag{14}$$

$$U_2(a, a) \geqslant 0. \tag{15}$$

The incentive compatibility constraints for buyer 2 together with the allocation rules completely pin down buyer 2's expected payment. Note that buyer 2's informational rent (i.e., payoff $U_2(v_2, v_2)$) is increasing in his valuation. Therefore, buyer 2's participation constraint needs to be satisfied only for the lowest type; i.e., $U_2(a, a) \geqslant 0$. In the optimal mechanism, this participation constraint for the lowest type will be binding; i.e., $U_2(a, a) = 0$. If $U_2(a, a) > 0$, the seller then again can increase her revenue by reducing $U_2(a, a)$.

5. The seller's optimization problem

The seller's objective is to maximizes the expected monetary transfers from the two buyers by picking $t_1(v_2)$, $t_2(v_2)$, $x_1(v_2)$, $x_2(v_2)$, $p_{11}(v_2)$ and $p_{21}(v_2)$, subject to the incentive compatibility constraints, the participation constraints, and the feasibility constraints. From the analysis in the last section, monetary transfers can be completely determined by the allocation rules and the recommendations. By using Eqs. (9) and (14), we have

$$\begin{aligned} R &= \int_a^b t_1(v_2)\, dF(v_2) + \int_a^b t_2(v_2)\, dF(v_2) \\ &= \int_a^b \mathbb{E}_{p_{11}(\cdot), p_{21}(\cdot)} \Big\{ x_1(v_2)\big\{\lambda_1\big[v_1 I_{\{v_2 < p_{11}(v_2)\}} + p_{11}(v_2) I_{\{v_2 \geqslant p_{11}(v_2)\}}\big] + (1-\lambda_1)v_1\big\} \end{aligned}$$

$$
\begin{aligned}
&+ x_2(v_2)\big\{\lambda_2\big(v_1 - p_{21}(v_2)\big)I_{\{v_2 \leqslant p_{21}(v_2)\}}\big\} \\
&+ x_1(v_2)\big\{\lambda_1\big[v_2 - p_{11}(v_2)\big]I_{\{v_2 \geqslant p_{11}(v_2)\}} + (1-\lambda_1)(v_2 - v_1)I_{\{v_2 \geqslant v_1\}}\big\} \\
&+ x_2(v_2)\big\{(1-\lambda_2)[v_2 I_{\{v_2 > v_1\}} + v_1 I_{\{v_2 \leqslant v_1\}}] \\
&+ \lambda_2\big[p_{21}(v_2)I_{\{v_2 \leqslant p_{21}(v_2)\}} + v_2 I_{\{v_2 > p_{21}(v_2)\}}\big]\big\} \\
&- \frac{1-F(v_2)}{f(v_2)}\big\{x_1(v_2)\big[\lambda_1 I_{\{v_2 \geqslant p_{11}(v_2)\}} + (1-\lambda_1)I_{\{v_2 \geqslant v_1\}}\big]\big\} \\
&- \frac{1-F(v_2)}{f(v_2)}\big\{x_2(v_2)\big[(1-\lambda_2)I_{\{v_2 > v_1\}} + \lambda_2 I_{\{v_2 > p_{21}(v_2)\}}\big]\big\}\bigg\}\, dF(v_2) + U_1 + U_2(a,a)
\end{aligned}
$$

$$
\begin{aligned}
= \int_a^b & x_1(v_2)\mathbb{E}_{p_{11}(\cdot)}\big\{\lambda_1\big[J(v_2)I_{\{v_2 \geqslant p_{11}(v_2)\}} + v_1 I_{\{v_2 < p_{11}(v_2)\}}\big] \\
&+ (1-\lambda_1)\big[J(v_2)I_{\{v_2 \geqslant v_1\}} + v_1 I_{\{v_2 < v_1\}}\big]\big\}\, dF(v_2) \qquad (16) \\
&+ \int_a^b x_2(v_2)\mathbb{E}_{p_{21}(\cdot)}\big\{(1-\lambda_2)\big[J(v_2)I_{\{v_2 > v_1\}} + v_1 I_{\{v_2 \leqslant v_1\}}\big] \\
&+ \lambda_2\big[J(v_2)I_{\{v_2 > p_{21}(v_2)\}} + v_1 I_{\{v_2 \leqslant p_{21}(v_2)\}}\big]\big\}\, dF(v_2) + U_1 + U_2(a,a). \qquad (17)
\end{aligned}
$$

The above equation is intuitive. If the seller can prohibit the resale between the buyers, then when she allocates the object to a particular buyer, she gets the virtual valuation of that buyer. However, with resale, she gets the virtual valuation of the final winner instead. For example, if the seller allocates the object to buyer 1 (the publicly known buyer) in the initial market, then with probability λ_1 buyer 1 proposes the price in the resale market. If the proposed offer is higher than the valuation of buyer 2 (the regular buyer), then buyer 1 will be the final winner; otherwise, buyer 2 will be the final winner. These terms are captured in (16) of Eq. (17). Note that buyer 1's virtual valuation is v_1, as he has no private information.

It is generally impossible to explicitly characterize the incentive compatible price offers $p_{11}(v_2)$ and $p_{21}(v_2)$. Therefore, characterizing the optimal mechanism to maximize the seller's revenue is not straightforward. Our approach in this paper is to find an upper bound for the seller's revenue, and then construct a feasible mechanism generating this upper bound revenue. We can then conclude that this mechanism is optimal. On the other hand, this approach does not allow us to verify whether or not the optimal mechanism we obtained is unique.

To find the upper bound seller revenue and construct an optimal mechanism, we need the following four lemmas. Let v_2^* solves $J(v_2^*) = v_1$. This v_2^* is the critical valuation for the regular buyer such that his virtual valuation is exactly equal to the publicly known buyer's (virtual) valuation. We have the following lemma.

Lemma 7.

$$\mathbb{E}_{p_{11}(v_2)}\big\{J(v_2)I_{\{v_2 \geqslant p_{11}(v_2)\}} + v_1 I_{\{v_2 < p_{11}(v_2)\}}\big\} \leqslant J(v_2)I_{\{v_2 \geqslant v_2^*\}} + v_1 I_{\{v_2 < v_2^*\}}.$$

The inequality in the lemma follows directly from the fact that the right-hand side is equal to the maximum of $J(v_2)$ and v_1, and the left-hand side is equal to either $J(v_2)$ or v_1. This lemma implies that, in Case 11, it is always the best for the seller to conceal buyer 2's report by making a fully pooling recommendation $p_{11}(v_2) = v_2^*$, assuming that it satisfies the incentive compatibility constraint. Similarly, we have

Lemma 8.

$$\mathbb{E}_{p_{21}(v_2)}\big\{J(v_2)I_{\{v_2>p_{21}(v_2)\}}+v_1I_{\{v_2\leqslant p_{21}(v_2)\}}\big\}\leqslant J(v_2)I_{\{v_2>v_1\}}+v_1I_{\{v_2\leqslant v_1\}}.$$

This lemma implies that, in Case 21, it is always the best for the seller to conceal buyer 2's report by making a fully pooling recommendation $p_{21}(v_2)=v_1$, assuming that it satisfies the incentive compatibility constraint.

From the above two lemmas and inequalities (10) and (15), we have

$$\begin{aligned}
(17) &\leqslant \int_a^b x_1(v_2)\big\{\lambda_1\big[J(v_2)I_{\{v_2\geqslant v_2^*\}}+v_1I_{\{v_2<v_2^*\}}\big]+(1-\lambda_1)\big[J(v_2)I_{\{v_2\geqslant v_1\}}+v_1I_{\{v_2<v_1\}}\big]\big\}\\
&\quad+\int_a^b x_2(v_2)\big\{(1-\lambda_2)\big[J(v_2)I_{\{v_2>v_1\}}+v_1I_{\{v_2\leqslant v_1\}}\big]\\
&\quad+\lambda_2\big[J(v_2)I_{\{v_2>v_1\}}+v_1I_{\{v_2\leqslant v_1\}}\big]\big\}\\
&=\int_a^b x_1(v_2)\big\{\lambda_1\big[J(v_2)I_{\{v_2\geqslant v_2^*\}}+v_1I_{\{v_2<v_2^*\}}\big]+(1-\lambda_1)\big[J(v_2)I_{\{v_2\geqslant v_1\}}+v_1I_{\{v_2<v_1\}}\big]\big\}\\
&\quad+\int_a^b x_2(v_2)\big\{(1-\lambda_1)\big[J(v_2)I_{\{v_2>v_1\}}+v_1I_{\{v_2\leqslant v_1\}}\big]\\
&\quad+\lambda_1\big[J(v_2)I_{\{v_2>v_1\}}+v_1I_{\{v_2\leqslant v_1\}}\big]\big\}.
\end{aligned} \tag{18}$$

Similarly to Lemma 7, we have

Lemma 9.

$$J(v_2)I_{\{v_2\geqslant v_2^*\}}+v_1I_{\{v_2<v_2^*\}}\geqslant J(v_2)I_{\{v_2\geqslant v_1\}}+v_1I_{\{v_2<v_1\}}.$$

From this lemma, we obtain

$$\begin{aligned}
(18) &\leqslant \int_a^b \big[x_1(v_2)+x_2(v_2)\big]\\
&\quad\cdot\underbrace{\big\{\lambda_1\big[J(v_2)I_{\{v_2\geqslant v_2^*\}}+v_1I_{\{v_2<v_2^*\}}\big]+(1-\lambda_1)\big[J(v_2)I_{\{v_2\geqslant v_1\}}+v_1I_{\{v_2<v_1\}}\big]\big\}}_{H(v_2)}\,dF(v_2)\\
&\leqslant \int_a^b \max\big\{H(v_2),0\big\}\,dF(v_2).
\end{aligned} \tag{19}$$

Thus, the seller's revenue is bounded by the right-hand side of (19), which is denoted as the upper bound revenue. This upper bound revenue depends crucially on the function $H(v_2)$, which properties we now examine.

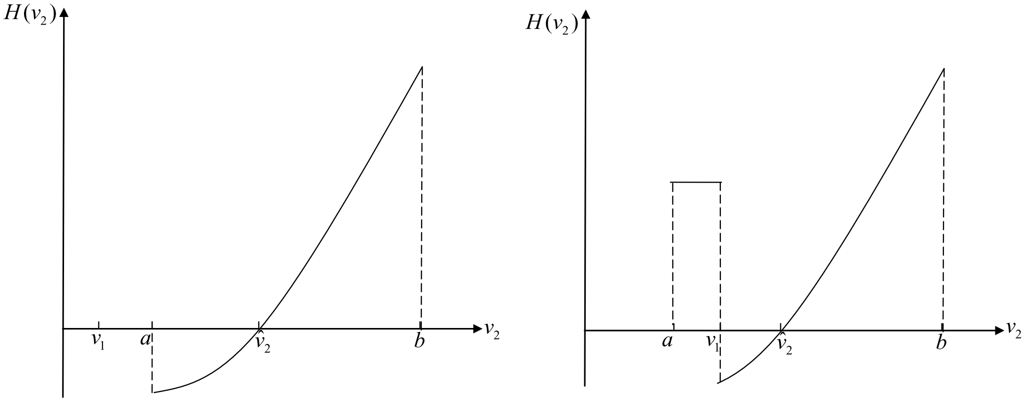

Fig. 1. Sketches of $H(v_2)$ in Situations 1 and 2.

Define $\hat{v}_2$ as the unique solution to $\lambda_1 v_1 + (1-\lambda_1)J(\hat{v}_2) = 0$ (if any). We have

Lemma 10. ***Situation* 1:** *$v_1 \leqslant a$ and $a < \hat{v}_2$. In this situation,*

$$H(v_2) = \begin{cases} \lambda_1 v_1 + (1-\lambda_1)J(v_2) < 0, & \text{if } a \leqslant v_2 < \hat{v}_2; \\ \lambda_1 \max\{J(v_2), v_1\} + (1-\lambda_1)J(v_2) \geqslant 0, & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases}$$

***Situation* 2:** *$a < v_1$ and $v_1 < \hat{v}_2$. In this situation,*

$$H(v_2) = \begin{cases} v_1 \geqslant 0, & \text{if } a \leqslant v_2 \leqslant v_1; \\ \lambda_1 v_1 + (1-\lambda_1)J(v_2) < 0, & \text{if } v_1 < v_2 < \hat{v}_2; \\ \lambda_1 \max\{J(v_2), v_1\} + (1-\lambda_1)J(v_2) \geqslant 0, & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases}$$

***Situation* 3:** *All other situations. $H(v_2) \geqslant 0$.*

Typical sketches for $H(v_2)$ in Situations 1 and 2 are given in Fig. 1. In Situation 1, $H(v_2)$ is increasing. It is negative for v_2 lower than $\hat{v}_2$ and positive for v_2 higher than $\hat{v}_2$. In Situation 2, $H(v_2)$ is positive except for an interval in the interior. In Situation 3, $H(v_2)$ is always positive.

If a feasible mechanism can generate the upper bound revenue, it would certainly be an optimal mechanism. It turns out that there always exists such a mechanism, which is formulated in the following theorem.

Theorem 1. *The following mechanism maximizes the seller's revenue*:

(i) *Allocation rules*: $x_2(v_2) = 0$ *and*

$$x_1(v_2) = \begin{cases} 0, & \text{if } H(v_2) < 0; \\ 1, & \text{if } H(v_2) \geqslant 0; \end{cases} \tag{20}$$

(ii) *Resale market price recommendations*: $p_{11}(v_2) = v_2^*$, $p_{21}(v_2)$ *can be any distribution independent of v_2, and $p_{12}(v_2)$, $p_{22}(v_2)$ are given by (4), (5), respectively.*

(iii) *Transfer payments to the seller*:
In Situations 1 and 2,

$$\mathbb{E}_{v_2} t_1(v_2) = \begin{cases} v_1[1-F(\hat{v}_2)] + \lambda_1[1-F(v_2^*)](v_2^* - v_1), & \text{if Situation } 1; \\ v_1[1+F(v_1)-F(\hat{v}_2)] + \lambda_1[1-F(v_2^*)](v_2^* - v_1), & \text{if Situation } 2; \end{cases}$$

$$t_2(v_2) = \begin{cases} 0, & \text{if } a \leqslant v_2 < \hat{v}_2; \\ (1-\lambda_1)(\hat{v}_2 - v_1), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases}$$

In Situation 3,

$$\mathbb{E}_{v_2} t_1(v_2) = v_1 + \lambda_1\big[1 - F\big(v_2^*\big)\big]\big(v_2^* - v_1\big),$$

$$t_2(v_2) = \begin{cases} (1-\lambda_1)(a - v_1), & \text{if } a \geqslant v_1; \\ 0, & \text{if } a < v_1. \end{cases}$$

(iv) *The seller's revenue is equal to* $\int_a^b \max\{H(v_2), 0\}\, dF(v_2)$.

This optimal mechanism has many properties, which will be discussed in details below. First, by examining the allocation rule, we have the following striking result.

Corollary 1. *In the above identified optimal mechanism, it is never optimal to allocate the object directly to the regular buyer* (*buyer* 2) *in the initial market.*

If the publicly known buyer has full bargaining power in the resale market, this result is intuitive; the seller can achieve the Myerson revenue by always allocating the object to the publicly known buyer. It is striking that the result remains valid even when the publicly known buyer has less than full bargaining power. If we interpret the publicly known buyer as a dealer, this result simply says that the seller would not sell the object to anyone but the dealer in this optimal mechanism. This is consistent with our observations. In a used car auction, for example, the seller (wholesaler) usually only allows dealers to bid, and is reluctant to deal with individual buyers.

One may think that the above intuition is from the fact that the publicly known buyer never earns any rents. By not allocating the object to the regular buyer in the original auction, his informational rent is minimized. However, assigning the object to different buyer will result in different bargaining situation in the resale market and will induce different total surplus to be extracted. For the seller, it is always better to generate a final allocation as close to the Myerson allocation as possible because of the Revenue Equivalence Theorem. The Myerson mechanism favors the publicly known buyer (buyer 1) in the sense that he wins more often compared to the efficient allocation.

The intuition for this corollary is clearest in Situation 3. To see the optimality of always allocating the object to the publicly known buyer, first suppose that the seller allocates the object to the regular buyer in the initial market. If the regular buyer is picked to propose the offer, then the final allocation is efficient. If the publicly known buyer is picked to propose the offer, then he will not propose an offer greater than his own valuation. As a result, the final allocation will favor the regular buyer and the seller's expected revenue is even less than the efficient allocation. Therefore, allocating the object to the regular buyer is even dominated by the efficient allocation.

Now suppose that the seller initially allocates the object to the publicly known buyer. If the regular buyer is picked to propose the offer in the resale market, then the final allocation is efficient. If the publicly known buyer is picked to propose the offer, then he will not propose an offer less than his own valuation. As a result, the final allocation will favor the publicly known buyer. In the mechanism we characterized in the above theorem, while the publicly known buyer may update his belief about the regular buyer's valuation in various ways, his updated belief is a truncation (either from below or in the middle) of the original distribution and includes all values in the interval $[v_2^*, b]$. Therefore, the updated virtual valuation function of the regular buyer remains the same from v_2^* and the publicly known buyer's optimal resale price is always equal to v_2^*. This final allocation coincides with the Myerson allocation. Therefore, in the initial market, allocating the object to the publicly known buyer dominates the efficient allocation, which in turn dominates the allocation of assigning the object to the regular buyer. Hence, it is never optimal to allocate the object to the regular buyer in the initial market.

Next, we examine the monetary transfers. The publicly known buyer can guarantee a benefit of v_1 if he is awarded the object. Moreover, he can obtain the extra benefit of $v_2^* - v_1$ if he happens to be the proposer in the resale market. Not surprisingly, all of these expected benefits will be exploited by the seller by setting the publicly known buyer's expected payment equal to his expected benefit from participation. The publicly known buyer does not have private information and thus obtains no informational rent.

Note that even though the seller does not allocate the object to the regular buyer in the initial market, she nevertheless demands some payment from the regular buyer if she decides not to retain the object. Even so, the regular buyer earn a positive payoff, and it is higher than the payoff when resale is prohibited. This suggests that the existence of hidden actions (because the resale market is not controlled by the seller) has two effects. First, for the buyer with private information (i.e., the regular buyer), the hidden action issue adds to the adverse selection problem and worsens the seller's revenue. She has to give the regular buyer more informational rents. Second, for the publicly known buyer who has no private information, he does not obtained any benefit from the existence of hidden actions. Therefore, the existence of hidden actions harms the seller only when it joins force with hidden information. Indeed, if both buyers' valuations are common knowledge, then it does not matter to the seller whether resale is prohibited or not. In either situation, the seller's revenue is the same and it is equal to the maximal of the two valuations.

The monetary transfers do not need to be collected at the end of the resale market in the above optimal mechanism. If we set the publicly known buyer's payment always equal to his expected monetary transfers, then this payment does not depend on v_2. Therefore, the payment would not reveal the regular buyer's valuation. Given this, the payment can be demanded from the publicly known buyer in the initial market. Since the regular buyer is the only one who has the private information, the seller can demand the payment from him at any time after he reports his valuation, as long as the exact payment amount is concealed from the publicly known buyer.

When resale is not prohibited, the seller cannot generate more than the Myerson revenue. This is because the seller's revenue depends only on the final allocation of the object, and the seller has full control of the final allocation in the Myerson mechanism. Thus, the Myerson revenue establishes another upper bound for the seller's revenue when resale is allowed. However, this upper bound may or may not be achievable. This is in contrast to the upper bound defined by the right-hand side of (19), which is always achieved in our optimal mechanism. Meanwhile, we can easily find a lower bound for the seller's maximum revenue when resale is allowed. It is bounded below by the revenue from the efficient mechanism. The seller can guarantee at least this revenue by implementing the efficient mechanism in the initial market, since no further trade will occur in the resale market given the allocation.

In what follows, we use a series of corollaries to illustrate other important properties of our optimal mechanism. The first one considers the seller's revenue when she does not retain the object in the optimal mechanism. Let R_M denote the Myerson revenue. Let R_E denote the (optimal) fully efficient mechanism revenue, where the buyer with the higher valuation always wins the object and both the publicly known buyer and the lowest valuation regular buyer get zero payoff. We have

Corollary 2. *In Situation* 3*, the seller's revenue in the optimal mechanism characterized by Theorem* 1 *is* $R = \lambda_1 R_M + (1 - \lambda_1) R_E$*. That is, the seller's maximum revenue is an average of the Myerson revenue and the fully efficient revenue weighted by the bargaining power of the publicly known buyer* (λ_1) *and the bargaining power of the regular buyer* ($1 - \lambda_1$)*.*

This result can be best understood from (19). In $H(v_2)$, $J(v_2)I_{\{v_2 \geqslant v_2^*\}} + v_1 I_{\{v_2 < v_2^*\}}$ is the virtual valuation in the Myerson mechanism and $J(v_2)I_{\{v_2 \geqslant v_1\}} + v_1 I_{\{v_2 < v_1\}}$ is the virtual valuation in the optimal efficient mechanism (where the buyer with the higher valuation wins). In the optimal mechanism characterized by Theorem 1, the right-hand side of (19) is achieved and the two inequalities become binding. In addition, in Situation 3, the $H(v_2)$ function is always positive. Therefore, the revenue of this optimal mechanism is the average of the two mechanisms weighted by the two buyers' respective bargaining powers.

Note that the seller always assigns the object to the publicly known buyer in Situation 3. If the publicly known buyer proposes the price in resale market (with probability λ_1), then the seller can induce the Myerson allocation as the final allocation by recommending the publicly known buyer to ask a price of $J^{-1}(v_1)$, and obtains revenue R_M; if the regular buyer proposes the price (with probability $1 - \lambda_1$), he will offer v_1 for sure, if profitable, as the publicly known buyer's valuation is common knowledge, and the final allocation will be efficient, providing the seller with revenue R_E.

Note also that $R_M \geqslant R_E$. Therefore, the seller' revenue is increasing in λ_1 in Situation 3. A similar relationship between the seller's revenue and the publicly known buyer's bargaining power holds in the other two situations as well, even though a simple explicit formula similarly to the one in the above corollary is not available. We have the following corollary.

Corollary 3. *The seller's revenue is increasing in the publicly known buyer's bargaining power* λ_1 *in all three situations.*

The intuition behind this corollary can be seen from the fact that the Myerson allocation generates the highest revenue for the seller. When the publicly known buyer has more bargaining power, the final allocation moves closer to the Myerson allocation, and therefore the seller's revenue is higher.

We mentioned earlier that the Myerson revenue may or may not be achievable. In the case where $v_1 = 0$, it can be easily verified that the Myerson revenue can be achieved by, besides the mechanism we constructed, completely shutting down the publicly known buyer. This is because the publicly known buyer has the same valuation as the seller. However, whenever the publicly known buyer's valuation is higher than the seller's reservation value, the following corollary shows that the Myerson revenue is almost never achievable.

Corollary 4. *When* $v_1 > 0$*, the optimal mechanism in Theorem* 1 *achieves the Myerson revenue if and only if the publicly known buyer has full bargaining power, i.e., if and only if* $\lambda_1 = 1$.

The intuition for this result is relatively straightforward. Obviously, the Myerson revenue cannot be achieved by simply implementing the Myerson allocation in the initial market. This is because buyers will trade further in the resale market, and such trading distorts the final allocation away from the Myerson allocation. However, when the publicly known buyer has full bargaining power (in the case he wins, i.e., $\lambda_1 = 1$), the seller can do the following to generate the Myerson revenue. The seller uses the publicly known buyer as a middleman by always allocating the object to him, and reveals no information (regarding the regular buyer's valuation) to the resale market. As a result, the publicly known buyer's belief about the regular buyer's valuation remains unchanged. Therefore, the publicly known buyer offers a price equal to the regular buyer's virtual valuation (evaluated at the publicly known buyer's valuation) to the regular buyer in the resale market. In this case, the final allocation coincides with the Myerson allocation. The Revenue Equivalence Theorem implies that this mechanism achieves the Myerson revenue. It means that the Myerson revenue is achievable even if the seller does not have full controlling power over the resale market. This result can be regarded as a special case of Zheng [15].

When $\lambda_1 < 1$, the seller's optimal revenue becomes strictly less than the Myerson revenue. Theorem 1 completely characterizes an optimal mechanism in this case, and the Myerson revenue is not attainable. When $\lambda_1 = 0$, surprisingly, the seller can generate no more than the lower bound revenue, i.e., the revenue from the fully efficient mechanism. We have

Corollary 5. *The fully efficient mechanism is optimal if and only if the publicly known buyer has no bargaining power* (*i.e.,* $\lambda_1 = 0$) *in the resale market and* $J(max\{a, v_1\}) \geqslant 0$.

When the condition $J(max\{a, v_1\}) \geqslant 0$ does not hold, the optimal mechanism is a conditionally efficient mechanism in the sense that the allocation is efficient among the buyers but the seller may retain the object inefficiently. In the optimal mechanism characterized by Theorem 1, this revenue is achieved by allocating the object only to the publicly known buyer and letting the buyers trade in the resale market. Alternatively, this revenue can be achieved by implementing the efficient allocation directly in the initial market, and no further trade will occur in the resale market. This illustrates that the optimal mechanism we constructed is not the unique optimal mechanism at least in certain situations.

Now we consider the information revealed to the resale market by the seller through the object allocations, the transfer payments, and the recommendations. As is evidenced in Theorem 1, all recommendations to the publicly known buyer do not depend on the regular buyer's reported valuation. We have the following corollary.

Corollary 6. *In the optimal mechanism constructed in Theorem* 1, *the seller does not reveal any additional information regarding the regular buyer's reported valuation* (*other than who wins and who loses*) *to the resale market. Revealing additional information may reduce the seller's revenue.*

If the seller conceals all information regarding the regular buyer's reported valuation, the publicly known buyer would set a price equal to the cutoff leading to the Myerson allocation when he is picked to make the offer. Any additional information will prompt the publicly known buyer to update his belief, and the price he offers may then roll away from the Myerson cutoff. If the final allocation is different from the Myerson allocation, then the seller cannot obtain the Myerson revenue. If the final allocation coincides with the Myerson allocation, on the other hand, then the seller still obtains the Myerson revenue. For example, in Corollary 5, implementing the fully efficient allocation in the initial market gives the seller the optimal revenue. In this case, even if some private information regarding the regular buyer's valuation is revealed to the resale market, it does not change the final allocation as no resale will occur.

We next consider the possibility of the seller retaining the object. Given the setup of our model, the seller does not retain the object in the Myerson mechanism, because the publicly known buyer's virtual value is always positive. However, the seller in the optimal mechanism characterized by Theorem 1 may find it optimal to retain the object under certain circumstances. Lemma 10 provides the conditions under which it is optimal for the seller to retain the object. We have the following corollary.

Corollary 7. *In Situation* 1*, it is optimal for the seller to retain the object if* $a \leqslant v_2 < \hat{v}_2$*. In Situation* 2*, it is optimal for the seller to retain the object if* $v_1 \leqslant v_2 < \hat{v}_2$*. In Situation* 3*, it is not optimal for the seller to retain the object. The probability of the seller retaining the object is decreasing in the publicly known buyer's bargaining power* λ_1*.*

The intuition is as follows. With resale, the seller obtains the virtual valuation of the final owner. The condition in the corollary reflects the situation where the regular buyer would be the final owner and where the regular's virtual valuation would be negative if the seller does not retain the object. When λ_1 decreases, the revenue moves towards the conditionally efficient allocation, and therefore, the seller should retain the object more often.

This corollary partially answers the question of whether allowing resale can improve the overall efficiency of selling. When resale is prohibited, the Myerson allocation is optimal; there is an efficiency loss because the mechanism overly favors the publicly known buyer. Meanwhile, the seller does not retain the object inefficiently. When resale is allowed, however, although resale induces a more efficient allocation between the two buyers, there is an efficiency loss from the seller inefficiently retaining the object. Therefore, allowing resale may not necessarily improve the overall efficiency.

Finally, suppose that the publicly known buyer is a pure speculator (with valuation v_1 equal to zero) as in Garratt and Troger [4]. We have the following corollary.

Corollary 8. *If* $v_1 = 0$*, then the optimal revenue can also be achieved by excluding the publicly known buyer from the object allocation.*

The publicly known buyer plays an important role in the optimal mechanism we characterized, as the seller always allocates the object to him whenever the object is not retained. When the publicly known buyer values the object at zero and becomes a pure speculator, however, the seller can also obtain the optimal revenue without the help of the publicly known buyer. The seller can simply make an (optimal) take-or-leave-it offer to the regular buyer directly; no resale will occur since the publicly known buyer has a valuation of zero. (Note that in Garratt and Troger [4], a pure speculator can still play an active role in the standard auctions with resale.) Of course, when the publicly known buyer's valuation is non-zero, his role is necessary for the seller to obtain the optimal revenue. Excluding him from the initial allocation (i.e., $x_1(v_2) \equiv 0$) yields a lower seller revenue, because the resale market will not achieve the optimal allocation through bargaining.

6. Conclusion and discussion

In this paper, we construct an optimal mechanism in an environment in which a seller is selling an indivisible object to two buyers. One buyer is a "regular" buyer with a continuous valuation distribution, and the other is a "publicly known buyer" whose valuation is fixed and known. We focus on the case where the seller cannot prohibit the resale of the object. Following Calzolari and Pavan [2], we model the resale market as an ultimatum bargaining game, with nature picking a proposer randomly to empower each of the buyers with some bargaining strength in the resale market. In this environment, the most striking result is that it is never optimal to assign the object to the regular buyer in the initial market. We also find that when the publicly known buyer's valuation is not zero, the revenue in Myerson's optimal auction can be achieved only when the winner of the initial market has full bargaining power. Furthermore, the seller's revenue is increasing in the winner's bargaining power. Meanwhile, the original seller retains the object more often than in Myerson's optimal auction, as long as the winner does not have full bargaining power. The probability of retaining the object is higher and higher when the winner has less and less bargaining power.

In the analysis, we show that the existence of the publicly known buyer is very important to the seller. Excluding the publicly known buyer from the mechanism usually reduces the seller's revenue unless he is a pure speculator with valuation zero. In the optimal mechanism, the role of the publicly known buyer is a middleman. As long as it is in the seller's interest to sell the object, she should always sell it to the publicly known buyer.

One natural extension is to allow some uncertainty on the publicly known buyer's valuation as well. If the valuation of this publicly known player has some small uncertainty, the mechanism in our paper may not be optimal, but the seller revenue would be close to the optimal value. So in this restricted sense, our mechanism is robust. This means that a buyer with a much higher level of uncertainty in his valuation is much less likely to be assigned the object. However, if we replace the publicly known player by another (probably asymmetric) regular buyer, then the analysis in this paper is inadequate, and characterizing the optimal mechanism would be an interesting future research project. Of course, our analysis does provide many insights. Without resale, buyers are ranked by their virtual valuations. In contrast, in our simple model with resale, buyers are ranked by the degree of uncertainty in their valuations, regardless of their bargaining power in the resale market. This suggests that in a general model with resale, both the virtual valuations and the dispersions of valuations are important in determining who wins the object in the initial market. Hopefully, the properties of the optimal mechanism constructed in this paper could be helpful in constructing the optimal mechanisms in the more general environments.

Another natural extension is to allow for more than two buyers. The challenge, as illustrated by Hafalir and Krishna [5], is that there are many modeling choices concerning the bargaining in the resale market. Nevertheless, given enough symmetry in the model, we can still apply the analysis in the paper to obtain an optimal mechanism.

Suppose that there are N *ex-ante* identical regular buyers whose valuations are drawn independently from distribution $F(\cdot)$ and, as before, one publicly known buyer. If the roles of these regular buyers are also identical in the resale market, the analysis in our paper can be applied in a straightforward way. For example, we can model the resale market as follows. If the publicly known buyer obtains the object in the initial market, then with probability λ_P he runs a second price auction with an optimal reserve price; with probability $1 - \lambda_P$, he runs a second price auction with a reserve price equal to his own valuation.[12] If one of the regular buyers wins the object in the initial market, the following stochastic ultimatum game takes place between this regular buyer and the publicly known buyer: with probability λ_R the regular buyer makes a take-it-or-leave-it offer; with probability $1 - \lambda_R$, the publicly known buyer makes a take-it-or-leave-it offer. Let λ_R be the same for each regular buyer to maintain symmetry. Note that when $N = 1$, this setup is identical to the model in our paper.

[12] The publicly known buyer has full bargaining power in the former situation and designs an optimal mechanism to resell the object. The publicly known buyer has no bargaining power in the latter situation and can only choose the best offer from those regular buyers.

To analyze this multiple regular buyer case, we first consider allocating the object to a regular buyer whose valuation is not the highest among those regular buyers. This allocation rule is dominated by the rule of allocating the object to the regular buyer with the highest valuation (while keeping all other allocation rules unchanged). This is simply because there is more revenue to share in the resale market in this way; from the point view of the seller, it is never optimal to allocate the object to a regular buyer whose valuation is not the highest among the regular buyers. Therefore, we can group all regular buyers as one player with distribution $F^N(\cdot)$. The seller is effectively gaming against two buyers: the publicly known buyer and a "new" regular buyer with distribution $F^N(\cdot)$. All of the results in our paper can then be applied directly. The seller will always allocate the object to the publicly known buyer and reveal no information to the resale market. If the publicly known buyer has full bargaining power, the Myerson allocation emerges; if the "new" regular buyer has full bargaining power, a "conditionally" efficient allocation occurs.

Appendix A

Proof for Lemma 6. From the Envelope Theorem, we have

$$\begin{aligned}\frac{dU_2(v_2, v_2)}{dv_2} &= \mathbb{E}_{p_{11}(\cdot),p_{21}(\cdot)}\big\{x_1(v_2)\big\{\lambda_1 I_{\{v_2 \geqslant p_{11}(v_2)\}} + (1-\lambda_1)I_{\{v_2 \geqslant v_1\}}\big\} \\ &\quad + x_2(v_2)\big\{(1-\lambda_2)[I_{\{v_2>v_1\}}] + \lambda_2[I_{\{v_2>p_{21}(v_2)\}}]\big\}\big\}. \end{aligned} \tag{21}$$

Solving the above differential equation gives us

$$\begin{aligned}U_2(v_2, v_2) &= \int_a^{v_2} \mathbb{E}_{p_{11}(\cdot),p_{21}(\cdot)}\big\{x_1(\xi)\big[\lambda_1 I_{\{\xi \geqslant p_{11}(\xi)\}} + (1-\lambda_1)I_{\{\xi \geqslant v_1\}}\big] \\ &\quad + x_2(\xi)\big[(1-\lambda_2)I_{\{\xi>v_1\}} + \lambda_2 I_{\{\xi>p_{21}(\xi)\}}\big]\big\}\,d\xi + U_2(a,a). \end{aligned} \tag{22}$$

Substituting Eq. (22) into Eq. (11) and setting $\tilde{v}_2 = v_2$ yields the desired result. □

Proof for Lemma 7. When $v_2 \geqslant v_2^*$, we have $J(v_2) \geqslant v_1$ since the virtual valuation $J(\cdot)$ is increasing. Therefore,

$$\begin{aligned} LHS &\leqslant \mathbb{E}_{p_{11}(\cdot)}\big\{J(v_2)I_{\{v_2 \geqslant p_{11}(v_2)\}} + J(v_2)I_{\{v_2<p_{11}(v_2)\}}\big\} \\ &= J(v_2)\mathbb{E}_{p_{11}(\cdot)}\{I_{\{v_2 \geqslant p_{11}(v_2)\}} + I_{\{v_2<p_{11}(v_2)\}}\} \\ &= J(v_2) = RHS. \end{aligned} \tag{23}$$

When $v_2 < v_2^*$, we have $J(v_2) \leqslant v_1$ since the virtual valuation $J(\cdot)$ is increasing. Therefore,

$$\begin{aligned} LHS &\leqslant \mathbb{E}_{p_{11}(\cdot)}\{v_1 I_{\{v_2 \geqslant p_{11}(v_2)\}} + v_1 I_{\{v_2<p_{11}(v_2)\}}\} \\ &= v_1\mathbb{E}_{p_{11}(\cdot)}\{I_{\{v_2 \geqslant p_{11}(v_2)\}} + I_{\{v_2<p_{11}(v_2)\}}\} \\ &= v_1 = RHS. \qquad \square \end{aligned} \tag{24}$$

Proof for Lemma 8. Note that when buyer 2 wins the object and buyer 1 makes the offer, buyer 1 will not offer a price higher than his own valuation v_1. Therefore, $p_{21}(v_2) \leqslant v_1$.

When $v_2 \geqslant v_1$, we have $v_2 \geqslant p_{21}(v_2)$, and therefore,

$$\begin{aligned} LHS &= J(v_2), \\ RHS &= J(v_2). \end{aligned}$$

When $v_2 < v_1$, we have $J(v_2) < v_1$, and therefore,

$$LHS \leqslant \mathbb{E}_{p_{21}(\cdot)}\{v_1 I_{\{v_2 > p_{21}(v_2)\}} + v_1 I_{\{v_2 \leqslant p_{21}(v_2)\}}\} = v_1,$$
$$RHS = v_1. \qquad \square$$

Proof for Lemma 9. When $v_2 \geqslant v_2^*$, we have $J(v_2) \geqslant v_1$ since the virtual valuation $J(\cdot)$ is increasing. Therefore,

$$LHS = J(v_2),$$

and

$$RHS \leqslant J(v_2) I_{\{v_2 \geqslant v_1\}} + J(v_2) I_{\{v_2 < v_1\}} = J(v_2) = LHS.$$

When $v_2 < v_2^*$, we have $J(v_2) \leqslant v_1$ since the virtual valuation $J(\cdot)$ is increasing. Therefore,

$$LHS = v_1,$$
$$RHS \leqslant v_1 I_{\{v_2 \geqslant v_1\}} + v_1 I_{\{v_2 < v_1\}} = v_1 = LHS. \qquad \square$$

Proof for Lemma 10. Note that $v_2^* \geqslant v_1$. When $v_2 \leqslant v_1$, $H(v_2) = \lambda_1 v_1 + (1 - \lambda_1) v_1 \geqslant 0$.

When $v_2 \geqslant v_2^*$, $H(v_2) = \lambda_1 J(v_2) + (1 - \lambda_1) J(v_2) \geqslant J(v_2^*) = v_1 \geqslant 0$.

When $v_1 < v_2 < v_2^*$, $H(v_2) = \lambda_1 v_1 + (1 - \lambda_1) J(v_2)$. In this case, $H(v_2)$ is increasing in v_2 since $J(v_2)$ is increasing. The upper bound of $H(v_2)$ is $H(v_2^*) = v_1 \geqslant 0$. Thus, if $a \geqslant v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J(a) < 0$, then there exists a unique $\hat{v}_2 \in (v_1, v_2^*)$, such that $H(\hat{v}_2) = 0$. If $a < v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J(v_1) < 0$, then there exists a unique $\hat{v}_2 \in (a, v_2^*)$, such that $H(\hat{v}_2) = 0$. In all other cases, $a \geqslant v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J(a) \geqslant 0$ or $a < v_1$ and $\lambda_1 v_1 + (1 - \lambda_1) J(v_1) \geqslant 0$, implying $H(v_2) \geqslant 0$. Summarizing these cases gives us the lemma. $\square$

Proof for Theorem 1. We need to show that the described mechanism satisfies all ICs and PCs, and achieves the upper bound revenue. Since $p_{12}(v_2)$ and $p_{22}(v_2)$ are directly taken from the lemmas on the buyers' incentive compatible constraints, they already satisfy IC. In addition, since only buyer 1 can win in the initial market, Case 21 is off the equilibrium path, and therefore $p_{21}(v_2)$ is not relevant as long as it does not reveal information. Therefore, we only need to show that $p_{11}(v_2)$ satisfies IC and truthful reporting in the initial market is optimal.

Recall that buyer 1's maximization problem in Case 11 is

$$\max_{\tilde{p}} \quad \Pi_1 = v_1 G_{11}(\tilde{p}) + \tilde{p}\big[1 - G_{11}(\tilde{p})\big]. \tag{25}$$

The first order condition is

$$\frac{d\Pi_{11}}{d\tilde{p}} = v_1 g_{11}(\tilde{p}) - \tilde{p} g_{11}(\tilde{p}) + \big[1 - G_{11}(\tilde{p})\big] = 0$$
$$\Rightarrow \quad v_1 = \tilde{p} - \frac{1 - G_{11}(\tilde{p})}{g_{11}(\tilde{p})}. \tag{26}$$

To proceed, we will examine each of the three situations separately.

In Situation 1, the winner, if any, is always buyer 1 in the initial market and the recommendations are fully pooling at v_2^*. Thus, when buyer 1 chooses the price to offer in Case 11, he believes that buyer 2's valuation is above $\hat{v}_2$, i.e., $G_{11}(v_2) = \frac{F(v_2)-F(\hat{v}_2)}{1-F(\hat{v}_2)}$. Therefore, substituting $G_{11}(v_2)$ into the (25) and the FOC yields:

$$v_1 = \tilde{p} - \frac{1 - \frac{F(\tilde{p})-F(\hat{v}_2)}{1-F(\hat{v}_2)}}{\frac{f(\tilde{p})}{1-F(\hat{v}_2)}} = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})} = J(\tilde{p}). \tag{27}$$

Therefore, $\tilde{p} = v_2^*$. It is easy to show that $\tilde{p} = v_1^*$ is a global maxima. Thus, recommendation $p_{11}(v_2) = v_2^*$ is incentive compatible.

We now verify buyer 2's incentive compatibility constraint in the initial market. Substituting all the relevant functions into Eq. (11), we have

$$\begin{aligned} &U_2(v_2, \tilde{v}_2) \\ &= \begin{cases} 0, & \text{if } a \leqslant v_2 < \hat{v}_2; \\ \lambda_1[v_2 - v_2^*]I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - v_1) - (1-\lambda_1)(\hat{v}_2 - v_1); & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b; \end{cases} \\ &= \begin{cases} 0, & \text{if } a \leqslant v_2 < \hat{v}_2; \\ \lambda_1(v_2 - v_2^*)I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases} \end{aligned} \tag{28}$$

First consider $v_2 \leqslant \hat{v}_2$. Note that $v_2 \leqslant v_2^*$, since $\hat{v}_2 \leqslant v_2^*$. Truthful reporting by buyer 2 implies that $U_2(v_2, v_2) = 0$. If he deviates to any $\tilde{v}_2 \leqslant \hat{v}_2$, it gives him the same payoff of 0. If he deviates to $\tilde{v}_2 \geqslant \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda_1)(v_2 - \hat{v}_2) \leqslant 0.$$

Thus, buyer 2 has no incentive to deviate.

Now consider $v_2 \geqslant \hat{v}_2$. In this case, $v_2 \geqslant v_1$. Truthful reporting by buyer 2 implies that

$$U_2(v_2, v_2) = \lambda_1\big(v_2 - v_2^*\big)I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2) \geqslant 0. \tag{29}$$

If buyer 2 deviates to any $\tilde{v}_2 \geqslant \hat{v}_2$, then it give him the same payoff. If he deviates to $\tilde{v}_2 \leqslant \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate.

In Situation 2, the winner, if any, is always buyer 1 in the initial market and the recommendation is fully pooling at v_2^*. Thus, when buyer 1 chooses the price to offer in Case 11, he believes that buyer 2's valuation is always in $[a, v_1] \cup [\hat{v}_2, b]$, i.e.,

$$G_{21}(v_2) = \begin{cases} \frac{F(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & \text{if } a \leqslant v_2 \leqslant v_1; \\ \frac{F(v_1)}{1-F(\hat{v}_2)+F(v_1)}, & \text{if } v_1 < v_2 < \hat{v}_2; \\ \frac{F(v_2)-F(\hat{v}_2)+F(v_1)}{1-F(\hat{v}_2)+F(v_1)}, & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b; \end{cases}$$

and

$$g_{21}(v_2) = \begin{cases} \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & \text{if } a \leqslant v_2 \leqslant v_1; \\ 0, & \text{if } v_1 < v_2 < \hat{v}_2; \\ \frac{f(v_2)}{1-F(\hat{v}_2)+F(v_1)}, & \text{if } \hat{v}_2 \leqslant v_2 \leqslant v_1; \end{cases}$$

$$\frac{1-G_{21}(v_2)}{g_{21}(v_2)} = \begin{cases} \frac{1-F(\hat{v}_2)+F(v_1)-F(v_2)}{f(v_2)}, & \text{if } a \leqslant v_2 \leqslant v_1; \\ +\infty, & \text{if } v_1 < v_2 < \hat{v}_2; \\ \frac{1-F(v_2)}{f(v_2)}, & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases}$$

Thus, by replacing these functions, the FOC (26) determining the offer price becomes

$$v_1 = \begin{cases} J(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})}, & \text{if } a \leqslant v_2 \leqslant v_1; \\ -\infty, & \text{if } v_1 < v_2 < \hat{v}_2; \\ J(\tilde{p}), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases} \tag{30}$$

This equation has a unique solution. First, note that $J(a) \leqslant v_1 \leqslant b$, $J(v_2)$ increasing, $J(\hat{v}_2) = 0$, and $J(v_2^*) = v_1$. If $a \leqslant v_2 \leqslant v_1$, (*RHS* of (30)) $< J(\tilde{p}) < J(v_2^*) = v_1$, and therefore there is no solution to (30). If $v_1 < v_2 < \hat{v}_2$, obviously there is no solution to (30). If $\hat{v}_2 \leqslant v_2 \leqslant b$, there is a unique solution $\tilde{p} = v_2^*$. We now need to show that $\tilde{p} = v_2^*$ is a global maxima. First,

$$\begin{aligned} \frac{d\Pi_1}{d\tilde{p}} &= v_1 g_{21}(\tilde{p}) - \tilde{p} g(\tilde{p}) + \big[1 - G_{11}(\tilde{p})\big] \\ &= g_{21}(\tilde{p})\left[v_1 - \tilde{p} + \frac{1 - G_{21}(\tilde{p})}{g_{21}(\tilde{p})}\right] \\ &= \begin{cases} g_{21}(\tilde{p})[v_1 - J(\tilde{p}) - \frac{F(\hat{v}_2)-F(v_1)}{f(\tilde{p})}], & \text{if } a \leqslant v_2 \leqslant v_1; \\ 1 - G_{11}(\tilde{p}), & \text{if } v_1 < v_2 < \hat{v}_2; \\ v_1 - J(\tilde{p}), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases} \end{aligned} \tag{31}$$

Since $v_1 = J(v_2^*)$ and $J(\cdot)$ is increasing, $\frac{d\Pi_1}{d\tilde{p}} \geqslant 0$ if $\tilde{p} \leqslant v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \leqslant 0$ if $\tilde{p} \geqslant v_2^*$. Thus, it is indeed optimal to follow the recommendation and offer price v_2^*.

We now verify buyer 2's incentive compatibility constraint in the initial market. Substituting all relevant functions into Eq. (11), we have

$$\begin{aligned} &U_2(v_2, \tilde{v}_2) \\ &\quad = \begin{cases} 0, & \text{if } a \leqslant v_2 \leqslant v_1; \\ 0, & \text{if } v_1 \leqslant v_2 < \hat{v}_2; \\ \lambda_1[v_2 - v_2^*]I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - v_1) - (1-\lambda_1)(\hat{v}_2 - v_1), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b; \end{cases} \\ &\quad = \begin{cases} 0, & \text{if } a \leqslant v_2 < \hat{v}_2; \\ \lambda_1(v_2 - v_2^*)I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2), & \text{if } \hat{v}_2 \leqslant v_2 \leqslant b. \end{cases} \end{aligned} \tag{32}$$

First consider $v_2 \leqslant \hat{v}_2$. Note that $v_2 \leqslant v_2^*$, since $\hat{v}_2 \leqslant v_2^*$. Truthful reporting by buyer 2 implies that $U_2(v_2, v_2) = 0$. If buyer 2 deviates to any $\tilde{v}_2 \leqslant \hat{v}_2$, it gives him the same payoff of 0. If he deviates to $\tilde{v}_2 \geqslant \hat{v}_2$, then

$$U_2(v_2, \tilde{v}_2) = (1-\lambda_1)(v_2 - \hat{v}_2) \leqslant 0.$$

Thus, he has no incentive to deviate.

Now consider $v_2 \geqslant \hat{v}_2$. In this case, $v_2 \geqslant v_1$. Truthful reporting implies that

$$U_2(v_2, v_2) = \lambda_1\big(v_2 - v_2^*\big)I_{\{v_2 \geqslant v_2^*\}} + (1-\lambda_1)(v_2 - \hat{v}_2) \geqslant 0. \tag{33}$$

If he deviates to any $\tilde{v}_2 \geqslant \hat{v}_2$, he obtains the same payoff. If he deviates to $\tilde{v}_2 \leqslant \hat{v}_2$, then $U_2(v_2, \tilde{v}_2) = 0$. Thus he has no incentive to deviate.

In Situation 3, since only buyer 1 can be the winner in the initial market and the recommendation is fully pooling at v_2^*, buyer 1 receives no additional information about buyer 2's valuation, i.e., buyer 1's valuation remains: $G_{11}(v_2) = F(v_2)$. Thus, by substituting $G_{11}(v_2) = F(v_2)$, the FOC (26) determining the offer price becomes $v_1 = \tilde{p} - \frac{1-F(\tilde{p})}{f(\tilde{p})} = J(\tilde{p})$. Since we assumed that

$J(a) \leqslant v_1 \leqslant b$ and that $J(v_2)$ increasing, there exists a unique solution $\tilde{p} = v_2^*$. We now show that $\tilde{p} = v_2^*$ is a global maxima. First,

$$\begin{aligned} \frac{d\Pi_1}{d\tilde{p}} &= v_1 f(\tilde{p}) - \tilde{p} f(\tilde{p}) + \left[1 - F(\tilde{p})\right] \\ &= f(\tilde{p})\left[v_1 - \tilde{p} + \frac{1 - F(\tilde{p})}{f(\tilde{p})}\right] \\ &= f(\tilde{p})\left[v_1 - J(\tilde{p})\right]. \end{aligned} \tag{34}$$

Since $v_1 = J(v_2^*)$ and $J(\cdot)$ is increasing, $\frac{d\Pi_1}{d\tilde{p}} \geqslant 0$ if $\tilde{p} \leqslant v_2^*$ and $\frac{d\Pi_1}{d\tilde{p}} \leqslant 0$ if $\tilde{p} \geqslant v_2^*$. Thus, it is indeed optimal for buyer 1 to follow the recommendation and offer price v_2^*.

We now verify buyer 2's incentive compatibility constraint in the initial market. Substituting all relevant functions into Eq. (11), we have

$$\begin{aligned} U_2(v_2, \tilde{v}_2) = {} & \left\{\lambda_1\left[v_2 - v_2^*\right] I_{\{v_2 \geqslant v_2^*\}} + (1 - \lambda_1)(v_2 - v_1) I_{\{v_2 \geqslant v_1\}}\right\} \\ & - \max\left\{(1 - \lambda_1)(a - v_1), 0\right\}. \end{aligned} \tag{35}$$

This payoff function does not depend on $\tilde{v}_2$, and therefore, buyer 2 has no incentive to lie about his valuation.

Finally, it can be easily shown that substituting the monetary transfer functions into the revenue function yields the upper bound revenue. □

Proof for Corollary 3. For Situation 1, the seller's revenue is

$$\begin{aligned} R &= \int_{\hat{v}_2}^{v_2^*} \left[\lambda_1 v_1 + (1 - \lambda_1) J(v_2)\right] dF(v_2) + \int_{v_2^*}^{b} \left[\lambda_1 J(v_2) + (1 - \lambda_1) J(v_2)\right] dF(v_2) \\ &= \int_{\hat{v}_2}^{v_2^*} \left[\lambda_1 v_1 + (1 - \lambda_1) J(v_2)\right] dF(v_2) + \int_{v_2^*}^{b} J(v_2)\, dF(v_2). \end{aligned} \tag{36}$$

Thus,

$$\begin{aligned} \frac{dR}{d\lambda_1} &= -\frac{d\hat{v}_2}{d\lambda_1}\left[\lambda_1 v_1 + (1 - \lambda_1) J(\hat{v}_2)\right] f(v_2) + \int_{\hat{v}_2}^{v_2^*} \left[v_1 - J(v_2)\right] dF(v_2) \\ &= \int_{\hat{v}_2}^{v_2^*} \left[v_1 - J(v_2)\right] dF(v_2) \geqslant 0. \end{aligned} \tag{37}$$

The last equality above follows from the definition of $\hat{v}_2$ and the inequality follows from the assumption that $J(v_2)$ is increasing and that $J(v_2^*) = v_1$.

For Situation 2,

$$R = \int_{a}^{v_1} \left[\lambda_1 v_1 + (1 - \lambda_1) v_1\right] dF(v_2) + \int_{\hat{v}_2}^{v_2^*} \left[\lambda_1 v_1 + (1 - \lambda_1) J(v_2)\right] dF(v_2)$$

$$+\int_{v_2^*}^{b}\big[\lambda_1 J(v_2)+(1-\lambda_1)J(v_2)\big]\,dF(v_2)$$

$$=v_1F(v_1)+\int_{\hat{v}_2}^{v_2^*}\big[\lambda_1 v_1+(1-\lambda_1)J(v_2)\big]\,dF(v_2)+\int_{v_2^*}^{b}J(v_2)\,dF(v_2). \tag{38}$$

Thus,

$$\frac{dR}{d\lambda_1}=-\frac{d\hat{v}_2}{d\lambda_1}\big[\lambda_1 v_1+(1-\lambda_1)J(\hat{v}_2)\big]f(v_2)+\int_{\hat{v}_2}^{v_2^*}\big[v_1-J(v_2)\big]\,dF(v_2)$$

$$=\int_{\hat{v}_2}^{v_2^*}\big[v_1-J(v_2)\big]\,dF(v_2)\geqslant 0. \tag{39}$$

For Situation 3,

$$R=\int_{a}^{v_2^*}\big[\lambda_1 v_1+(1-\lambda_1)J(v_2)\big]\,dF(v_2)+\int_{v_2^*}^{b}\big[\lambda_1 J(v_2)+(1-\lambda_1)J(v_2)\big]\,dF(v_2). \tag{40}$$

Thus,

$$\frac{dR}{d\lambda_1}=\int_{a}^{v_2^*}\big[v_1-J(v_2)\big]\,dF(v_2)\geqslant 0.$$

To summarize, since the revenue is a continuous function of λ_1, and in each case it is increasing in λ_1, it must be increasing in λ_1 over its entire domain. □

Proof for Corollary 7. Note that $\hat{v}_2$ is determined by $\lambda_1 v_1+(1-\lambda_1)J(\hat{v}_2)=0$. The immediate implication is that $J(\hat{v}_2)<0$. Differentiating both sides with respect to λ_1 yields:

$$v_1-J(\hat{v}_2)+(1-\lambda_1)J'(\hat{v}_2)\frac{\partial\hat{v}_2}{\partial\lambda_1}=0 \quad\Leftrightarrow\quad \frac{\partial\hat{v}_2}{\partial\lambda_1}=\frac{J(\hat{v}_2)-v_1}{(1-\lambda_1)J'(\hat{v}_2)}<0$$

Since the probability for the seller to retain the object is nondecreasing in $\hat{v}_2$ in all three situations, that probability is also decreasing in λ_1. □

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